MODULE 12: ARITHMETIC AS THE GATEWAY TO ALGEBRA

Suppose a geography teacher wanted to test to see whether the class knew that Sacramento was the capital of California. In terms of a fill-in-the-blank question, the instructor could ask either:

__________ is the capital of California. (1)

or

Sacramento is the capital of __________. (2)

Filled in correctly, both (1) and (2) would read:

Sacramento is the capital of California.

But there is a tremendous psychological difference between (1) and (2). In (1) we see "California" and when we think of California we are more likely to think of Los Angeles, San Diego, San Francisco, or Hollywood than we are to think of Sacramento. In (2), however, we see "Sacramento" and when we think of Sacramento we most likely will think of California.

In other words, the students will probably do better with (2) than with (1). They might say, "Gee, I didn't know that Sacramento was the capital of anything, but seeing that it's in California, I guess "california" is the answer to the question."

So while both questions test for the same information, the likelihood of getting the right answer depends on which name is replaced by the blank. *It is in exactly this same way that we may view the difference between arithmetic and algebra!* In terms of "fill-in-the-blank" arithmetic and algebra are two different ways of viewing the same piece of information.

Indeed, as we shall soon see, we've already done some algebra is this course!

For example, suppose an arithmetic teacher wants to test the class on the number fact that $3 + 8 = 11$.

The teacher could ask either:

$$3 + 8 = \underline{\quad\quad} \qquad (3)$$

or

$$3 + \underline{\quad\quad} = 11 \qquad (4)$$

Filled in correctly, both (3) and (4) would state that 3 + 8 = 11. However (3) looks like an addition problem, and it is an addition problem; *but while (4) may look like an addition problem, it is really a* <u>subtraction</u> *problem.* One way to help visualize the difference between (3) and (4) is in terms of a hand calculator. Suppose, for example, that you didn't know what addition meant but that you could press the appropriate buttons on the calculator. Given (3), you would simply apply the following steps.

(4) asks us to find the number we must add to 3 to get 11 as the sum. This number is represented by 11 - 3.

1. Press "3"
2. Press "+"
3. Press "8"
4. Press "="
5. The answer (11) now appears in the calculator's display.

All you're doing here is to simply press the symbols in "3 + 8 = ___" in the given order to get the answer. That is:

$$\underset{1}{3} \to \underset{2}{+} \to \underset{3}{8} \to \underset{4}{=}$$

But if you applied the same procedure to (4), you'd run into trouble when you got to the blank.

Roughly speaking we call it an arithmetic problem if we can get the correct answer just by entering the given symbols in the correct order into the calculator. If we can't do this, then we call the question an algebra *problem.*

In other words, if the blank stands by itself and the addition is only applied to the numbers, it is an arithmetic problem. But if one of the summands is the "blank" then it is an algebra problem.

Example 1

Fill in the blank:

$\_\_\_ \times 6 = 180$

Answer: 30

We're looking for the number we must multiply by 6 to get 180 as the product. That is, the number we want is $180 \div 6$ or 30.

We are not multiplying 180 by 6. If we were it would be the arithmetic problem:
$180 \times 6 = \_\_\_$

Key Point

$\_\_\_ \times 6 = 180$ (5)

and

$180 \div 6 = \_\_\_$ (6)

are different ways of asking for the same number. The difference is that (5) is an algebra problem, while (6) is an arithmetic problem.

(6) can be done on the hand calculator just by entering the symbols in the given order, but in (5) one of the factors is the blank--and we can't multiply a number by a blank on the calculator

Example 2

Rewrite $\_\_\_ \div 4 = 28$ as a fill-in-the-blank problem that can be done directly with the hand calculator.

Answer: $28 \times 4 = \_\_\_\_\_\_$

Note that by the commutative property of multiplication the answer can be written as $4 \times 28 = \_\_\_\_$

In the language of long division the problem is: $4\overline{)\,\_\_\_\,}$ with quotient 28

The answer is not 7. If we replace the blank by 7 we get the false statement:
$7 \div 4 = 28$

Recall that the check for division is that the product of the quotient (28) and the divisor (4) must equal the dividend (___). That is, $28 \times 4 = \_\_\_\_$

We can use the calculator to get $28 \times 4 = 112$ and thus fill in the blank, but this isn't what the example asks us to do.

In Example 2 we started with a problem that couldn't be done in the given form with the calculator and converted into an equivalent problem that could be

done directly on the calculator. *When we do this we are doing algebra.*

Rather than use blanks (which, among other things might be confused with minus signs) in algebra, we use letters of the alphabet. For example, rather than write $3 + 8 =$ ___, we might write $3 + 8 = n$. When we write $3 + 8 = n$, we obviously do not say "Fill in the blank". Instead we say "For what value of n is $\underline{3 + 8 = n}$ a true statement?" Sometimes this is abbreviated simply by the command "Find n" or "Solve for n".

"Fill in the blank" is itself an abbreviation for asking: "What word(s) must replace the blank to give us a true statement?"

Example 3

Solve for n if $11 \times 6 = n$.

Answer: $n = 66$

The n stands for a blank. This the question could have read: $11 \times 6 =$ _____. Since $11 \times 6 = 66$, we say that $n = 66$.

When we say $n = 66$ we mean that n stands for 66. That is, if we replace n by 66 $11 \times 6 = n$ becomes the true statement $11 \times 6 = 66$

To solve for n in Example 3, required only that we find the product of two numbers. This can be done directly on the calculator. Hence solving $11 \times 6 = n$ for n is basically an arithmetic problem even though the n may make it look like an algebra problem.

Some people like to call it "algebra" as soon as letters replace blanks. I prefer to think of $11 \times 6 = n$ as arithmetic problem because we can solve for n without having to know anything but the multiplication table.

Example 4

Solve for t if $15 + 7 = t$.

Answer: $t = 22$

In terms of a blank we have

$15 + 7 =$ ___

and since $15 + 7 = 22$, we write $t = 22$.

In other words if we replace t by 22 in $15 + 7 = t$ we get the true statement $15 + 7 = 22$.

Example 4 is only trying to emphasize that it is not important what letter we use to replace the blank. What is important is that we recognize that the letter stands for the blank. In a sense, both the letter and the blank are *"place holders"* that hold the place of the number that makes the statement true.

Instead of asking "Sacramento is the capital of ____", we could have asked, "Solve for n if 'Sacramento is the capital of n' is to be a true statement"

<u>Example 5</u>

Solve for q if $34 + q = 70$.

Answer: $q = 36$

If we replace q by a blank we have

$34 + \_\_\_\_ = 70$

In turn this tells us that we want the number we must add to 34 to get 70 as the sum. Since $34 + 36 = 70$, the answer is 36. *In other words, we must replace q by 36 in order that $34 + q = 70$ become a true statement.*

That is, $34 + \_\_\_ = 70$ is equivalent to $70 - 34 = \_\_\_\_$ This is what we were doing in problems such as Example 2

<u>*Note:*</u>

If you wrote the answer as $q = 104$ you solved the wrong problem. $q = 104$ is the answer to $34 + 70 = q$ which is quite different from $34 + q = 70$.

Example 5 illustrates what we mean by algebra. To solve $34 + q = 70$ directly by arithmetic, we rewrote the problem as $70 - 34 = q$. While q appears in both $34 + q = 70$ and $70 - 34 = q$, we call $34 + q = 70$ an algebra problem, but we call $70 - 34 = q$ an arithmetic problem.

In an equality it makes no difference which number we write first. For example we may write either $3 + 2 = 5$ or $5 = 3 + 2$. For this reason $70 - 34 = q$ is often written as $q = 70 - 34$.

Perhaps a little new vocabulary will make the idea clearer.

Some New Vocabulary

Any statement expressing the equality of two or more quantities is called an equation.

(1) If the equation involves only numbers, we call it a numerical equation. *For example, 3 + 8 = 11 is a numerical equation.*

(2) If an equation involves a letter but the letter is not used in the arithmetic, we call it an arithmetical equation. *For example, 3 + 8 = n or n = 3 + 8 are arithmetical equations.*

(3) If the letter is used as part of the arithmetic we call the equation an algebraic equation. *For example, 3 + n = 11 is called an algebraic equation.*

The word "equation" has the same origin as the word "equal"

3 + 8 = 11 is always a true statement. So we sometimes call it an equality. On the other hand, 3 + 8 = n is true only on the condition that n is 11. For this reason we call 3 + 8 = n a conditional equation.

3 + n = 11 is also true only under the condition that n is 8. So we call 3 + n = 11 a conditional equation also.

Example 6

Which of the following is an algebraic equation:

(a) 15 X 6 = 90
(b) 15 X m = 90
(c) 15 X 6 = m
(d) m = 15 X 6

Answer: (b)

To be an algebraic equation the equation has to involve a letter that is used in the arithmetic. This eliminates 15 X 6 = 90 because no letter are involved. It also eliminates both 15 X 6 = m and m = 15 X 6 because the only arithmetic involves multiplying the *number* 15 by the *number* 6.

15 X 6 = m and m = 15 X 6 mean exactly the same thing. In either case we solve for m by multiplying 15 and 6.

In (b), however, the multiplication involves the number 15 and the *letter* m. This is the requirement for an algebraic equation.

Equations, both arithmetical and algebraic, often arise from formulas. Recall, for example, the formula that relates the marked price to the total cost including tax if the tax is 5% of the marked price. In words, we have:

<u>Step 1</u>: Start with the marked price, which we'll denote by M.

We can use any letter but M seems to suggest <u>m</u>arked price.

<u>Step 2</u>: Multiply the number in Step 1 by 1.05.

Steps 1 and 2 may be abbreviated by M X 1.05

<u>Step 3</u>: The answer in Step 2 is the total price including the tax. We denote it by T.

*"is" translates as "="
That is, we may write the three steps as:
M X 1.05 = T or T = M X 1.05*

We chose T to suggest <u>t</u>otal cost.

The algebraic formula that represents Steps 1 through 3 is simply:

$$M \times 1.05 = T \qquad (7)$$

If we're given the value of M in (7) and want to find the corresponding value of T, we replace M by its given value and solve the resulting *arithmetical* equation.

If the value of M changes so does the value of T. For this reason M and T are called variables. *When one of the variables is replaced by a number, the formula becomes an equation, which can be solved for the other variable*

<u>Example 7</u>

Given the formula M X 1.05 = T, solve for T if M = 60.

Answer: If M = 60, then T = 63.

That is, T = 63 on the condition that M = 60.

Being told that M = 60 means that we are to replace M by 60 in the formula. If we do this we get:

$$\begin{array}{ccccc} M & X & 1.05 & = & T \\ \downarrow & & & & \\ 60 & X & 1.05 & = & T \end{array}$$

which is an arithmetical equation and its solution is found by multiplying 60 by 1.05.

That is, to find the value of T we need only multiply the numbers *60 and 1.05*

Since 60 X 1.05 = 63, the answer is T = 63.

In words, if we start with 60 and multiply by 1.05 the answer is 63.

1. *Start with M.......60*
2. *Multiply by 1.05*
3. *The answer is 60 X 1.05 or 63.*

However if we're given the value of T in (7) and are asked to find the corresponding value of M, we replace T by its given value, and solve the resulting *algebraic* equation.

Example 8

Given the formula M X 1.05 = T, solve for M if T = 84.

Answer: M = 80

In this case we start with M X 1.05 = T and replace T by 84 to get:

$$\begin{array}{ccccc} M & X & 1.05 & = & T \\ & & & & \downarrow \\ M & X & 1.05 & = & 84 \end{array} \qquad (8)$$

(8) is an algebraic equation because the *letter* M is involved in the arithmetic.

That is, M is being multiplied by 1.05.

Since the answer is 84 *after* we multiplied by 1.05, before we *multiplied* by 1.05 the amount (M) must have been 84 *divided by* 1.05.

This is another way of saying that ___ X 1.05 = 84 means the same as 84 ÷ 1.05 = ___

That is, we solve the algebraic equation M X 1.05 = 84 by converting it into the equivalent arithmetical equation M = 84 ÷ 1.05

We are now in a relatively good position to see where algebra fits into an arithmetic course.

(1) *The need for arithmetic arises when we are studying the relationship between one quantity (variable) and another quantity.*

For example, we might study the relationship between the marked price and the total cost.

(2) *We begin by expressing the relationship mathematically. This relationship is called a* formula.

For example, $M \times 1.05 = T$

(3) *Given the value of one of the variables, we put that value into the formula. This will result in a* conditional equation.

(4) *If the conditional equation is arithmetical we solve it by doing the type of arithmetic we learned in the first eleven modules of this course.*

$M \times 1.05 = T$
$\downarrow$
$6 \times 1.05 = T$, *so we simply multiply 6 by 1.05*

(5) *However if the resulting equation is algebraic then we have to convert it into an equivalent arithmetical equation which we then solve by one of the techniques studied in the first eleven modules of this course.*

$M \times 1.05 = T$
$\downarrow$
$M \times 1.05 = 21$ *is an algebraic equation that we convert into the equivalent arithmetical equation* $M = T \div 1.05$

What makes algebra difficult is that many algebraic equations are sufficiently complicated that it requires great computational skill as well as mastery of many computational devices to convert the equation into an equivalent arithmetical equation.

But the theory never changes. In algebra, we are always converting algebraic equations into arithmetical equations.

In this module we shall limit our study to a few of the less complicated techniques. In particular, we shall look at two general methods. Perhaps the best way to introduce these two methods is in terms of problems that we can already solve.

Example 9

The variables b and c are related by the formula $b = c + 4$. Find the value of c when $b = 9$.

Answer: When $b = 9$, $c = 5$

(That is $c = 5$ on the condition that $b = 9$)

Since we already have the formula that relates b and c, we replace b by 9 in this formula to get:

$$b = c + 4$$
$$\downarrow$$
$$9 = c + 4 \qquad (9)$$

This is an algebraic equation that can be converted into the equivalent arithmetical equation:

$$9 - 4 = c \qquad (10)$$

That is, c is the number which when added to 4 gives 9 as the sum.

To solve (10), all we have to know is the arithmetic fact that $9 - 4 = 5$.

Check: $b = c + 4$
$\downarrow \quad \downarrow$
$9 = 5 + 4$ *which is true*

To get from (9) to (10), the strategy is to express c by itself on one side of the equation and have all the numbers on the other side.

Note:

Numbers like 4 and 9 are called constants *because they remain the same throughout the problem. So our strategy is to get the variable by itself on one side of the equation and to have all the constants on the other side. Once we've done this we have by definition an arithmetical equation.*

That is, if we change either b or c in the formula $b = c + 4$, the other changes. But 4 remains the same. It is constant.

Looking at (9) we see that c appears only on the right hand side of the equation. But the constant 4 is also on that side. We want c to appear by itself which means that we want to "get rid of" the 4. Since the 4 is being *added* to c we "get rid of" it by *subtracting* 4 from the left side of the equation.

For example if you have $10 and add $4 you have $14. If you now subtract $4 from the $14 you have the original $10.

But an equation is like a balanced seesaw. Whatever you do on one side you have to do on the other if you want to maintain the balance. Hence if we subtract 4 from one side of equation (9) we also have to subtract 4 from the other side. If we do this we get:

$$\begin{array}{rcrr} 9 & = & c & +\ 4 \\ -\ 4 & & & -\ 4 \\ \hline 5 & = & c & \end{array}$$

So we got the 5 by computing 9 - 4, which is exactly what (10) told us to do.

Since the term "get rid of" is not too complimentary we shall refer to this process as the *"undoing" method.* The basic idea behind the undoing method may be summarized as follows:

(1) *To undo* adding *a number to one side of an equation, we* subtract *that number from both sides of the equation.*

(2) *To undo* subtracting *a number from one side of an equation, we* add *that number to both sides of the equation.*

(3) *To undo* multiplying *by a number on one side of an equation, we* divide *both sides of the equation by that number.*

Keep in mind, of course, that we are still not allowed to divide by 0.

(4) *To undo* dividing *one side of an equation by a number, we* multiply *both sides of the equation by that number.*

(5) *To undo* squaring *one side of an equation, we take the* square root *of both sides of the equation.*

Recall that squaring a quantity means to multiply that quantity by itself.

(6) *To undo taking the* square root *of one side of an equation, we* square *both sides of the equation.*

Let's practice.

Example 10

Use the undoing method to solve the equation $c - 7 = 11$.

We want c by itself on one side of the equation. It is already on the left hand

Answer: $c = 18$

Check:

$$\begin{array}{ccccc} c & - & 7 & = & 11 \\ \downarrow & & & & \\ 18 & - & 7 & = & 11 \end{array}$$

(true)

side. But 7 is also on the left hand side of the equation, and we want c to be alone.

Therefore we have to undo subtracting 7 from the left hand side of the equation. We do this by adding 7 to both sides of the equation and we get:

$$\begin{array}{rcl} c - 7 & = & 11 \\ +\ 7 & & +\ 7 \\ \hline c & = & 18 \end{array}$$

See? Adding 7 "cancelled" the 7 from the left side and increased the right side by 7. In essence, we converted the given algebraic equation into the arithmetical equation: c = 11 + 7

In using the undoing method it is not enough to see what number we want to undo. We must also see how this number is used in the arithmetic.

<u>Example 11</u>

Use the undoing method to solve the equation c X 7 = 56.

Answer: c = 8

Just as in Example 10, c already appears only on the left side of the equation and 7 is still on the left hand side with c. But this time c is being *multiplied* by 7. To undo multiplying by 7 on one side of an equation we have to divide both sides of the equation by 7. In this case we get:

The equation tells us that c is the number that we must multiply by 7 to get 56 as the product. This is the definition of 56 ÷ 7. However, the undoing method is showing us how to get this result in an automatic way.

$$(c \times 7) \div 7 = 56 \div 7$$

$$c = 56 \div 7 \quad (11)$$

$$c = 8$$

We're dividing the entire left hand side by 7. Since the left hand side is c X 7 we enclose it in parentheses to remind us of this fact.

Note that the undoing method converted the algebraic equation into (11) which is an arithmetical equation.

Check: $c \times 7 = 56$

$\downarrow$

$8 \times 7 = 56$ *(true)*

Some Special Notation in Algebra

The multiplication symbol X can be confused with the letter X. So to avoid this possible confusion in equations, we do not use the multiplication symbol X.

Rather than write, for example, 5 X c we write 5c. In other words, whenever a constant appears next to a variable without an arithmetic symbol between them, it is understood that the operation is multiplication.

If we had c X 5, we might write this as c5, but it is traditional to always write the constant factor, in the case 5, on the left. This is consistent with iur saying 5 apples rather than apples 5

Example 12

In the formula W = 8P, find the value of W when P = 4.

Answer: When P = 4, W = 32

W = 8P is an abbreviation for W = 8 X P.

If we replace P by 4, we get:

$$\begin{aligned} W &= 8 \text{ X } P \\ & \quad\quad\;\; \downarrow \\ W &= 8 \text{ X } 4 \end{aligned}$$

which is an arithmetical equation, from which it is easy to see that W = 32.

CAUTION

You must remember that 8P means we're multiplying 8 by P. You don't *want to replace P by 4 to get:*

$$\begin{aligned} W &= 8P \\ & \quad\;\; \downarrow \\ W &= 84 \end{aligned}$$

So it should be clear that when we multiply two constants we cannot simply write them side by side. If we did 8 X 4 would be confused with the place value numeral 84.

Some people prefer to use parentheses to avoid this problem. Rather than write 8P they write 8(P) and if we now replace P by 4 we get 8(4). The parentheses keeps 8(4) from looking like 84. Other people use • to stand for multiplication. Rather than write 8P they would write 8•P. Now when P is replaced by 4 we get 8•4. In this notation we must be careful not to confuse • with a decimal point. 8.4 is not the same as 8•4

Example 13

In the formula W = 8P, find the value of P when W = 72.

Answer: When W = 72, P = 9

In this case, we replace W by 72 to get:

$$
\begin{aligned}
W &= 8P \\
&\text{or} \\
W &= 8 \times P \\
\downarrow& \\
72 &= 8 \times P \qquad (12)
\end{aligned}
$$

(12) is an algebraic equation that is equivalent to the arithmetical equation

$$72 \div 8 = P$$

The solution of this equation is P = 9.

If you want to use the undoing method here, notice that 8 is multiplying P on the right side of the equation. To undo this, we have to divide both sides of the equation by 8. The important point is that we're trying to solve the equation:

72 = 8 X ____

Some of us are frightened by mathematical symbols. In this case, you might feel a bit tense with the symbolism involved in the undoing method. To help overcome this problem, there is another method that is more verbal.

For example, the formula c + 4 = b might look like a foreign language to you. To see how to read a formula, we look for the variable (letter) that is involved in the arithmetic. In this case 4 is being added to c. If the formula had been written in the form b = c + 4, we would still start with c because it is the letter that is involved in the arithmetic. *So to translate the formula into "English" we'd begin by saying "Start with c"*

That is, we don't want the variable that stands by itself on one side of the formula.

b = c + 4

↑

"Start with c"

Then we look to see what is being done to c. In this case we have c + 4, which we recognize as meaning that 4 is being added to c. So our second step in translating the formula would be to say "Add 4."

Start with c
$\downarrow$
b = c + 4
$\uparrow$ *Then add 4*

The equal sign (=) is read as "is" or "is equal to" So the final step would be to say "The answer is b"

Start with c (1)
$\downarrow$
b = c + 4
$\uparrow$ *Then add 4* (2)
The answer is b (3)

In summary b = c + 4 or equivalently c + 4 = b may be translated into:

Step 1: Start with c
Step 2: Add 4
Step 3: The answer is b

Example 14

Translate X = Y - 7 into a verbal formula.

Answer: Step 1: Start with Y
Step 2: Subtract 7
Step 3: The answer is X

We begin by looking for the variable that is involved in the arithmetic. Since X stands alone, the variable we start with is Y.

Y is on the right hand side of the equation. The only other number on that side is 7. Y - 7 tells us that 7 is being subtracted from Y. So our second step is "Subtract 7".

The other side of the equal side is X. So the last step is "The answer is X"

It may be more suggestive to rewrite the formula as
Y - 7 = X
This corresponds more to the natural order of the steps.

In this way the problem that asks "Given the formula X = Y - 7, find the value of X when Y = 15." can be reworded as: "Start with 15. Then subtract 7. Then write the answer."

That is:
(1) Start with Y (15)
(2) Subtract 7 (15 - 7)
(3) The answer is Y (8)

Using the verbal method gives us another way to visualize the undoing method. Namely, the undoing method means that we're starting with the answer and we want to know what number would yield the answer.

For example, in terms of Example 14, suppose we had:

<u>Step 1</u>: Start with a number (Y)
<u>Step 2</u>: Subtract 7 (Y - 7)
<u>Step 3</u>: The answer is 8 (X)

What number did we start with?

See how this compares with Example 14? In Example 14 we started with 15 and found that the answer was 8. Here we know that the answer is 8 and we want to find the number we started with.

In such an event, we begin with the *last* step and retrance our steps, making **sure** to "undo". That is we replace "add" by "subtract", "multiply" by "divide" and so on, until we get back to the original first step. *This process is known as* <u>inverting</u> *the formula.*

By way of illustration, to invert the formula X = Y - 15, we have:

<u>"Original Formula"</u>

(1) Start with Y → → → The answer is Y (3)
↓ ↑
(2) Subtract 7 → → → → Add 7 (2)
↓ ↑
(3) The answer is X → → Start with X (1)
↑
Inverse Formula

So in symbols the inverse formula becomes: Start with X, add 7 (X + 7), the answer is Y (X + 7 = Y or Y = X + 7)

In essense all we've done is:

$$\begin{array}{r} X = Y - 7 \\ +7 \qquad +7 \\ \hline X + 7 = Y \end{array}$$

While the algebraic way is much more concise, the verbal approach is often more intuitive, hence less threatening.

You **should** keep in mind that the verbal approach and the algebraic approach are different ways of representing the same ideas. So you should feel free to use whichever method is most comfortable for you. However, by understanding both approaches, you will have a better chance to really understand the meaning of a formula.

Example 15

Invert the formula:

Step 1: Start with q
Step 2: Divide by 4
Step 3: The answer is u

Answer:

Step 1: Start with u
Step 2: Multiply by 4
Step 3: The answer is q

We start at the last step and retrace our steps using the "undoing" language.

(1)	Start with q ↓	The answer is q ↑	(3)
(2)	Divide by 4 → → →	Multiply by 4 ↑	(2)
(3)	The answer is u → →	Start with u	(1)

The "undoing" step involves replacing "divide" by "multiply"

In the language of algebra the given formula is $q \div 4 = u$. Inverting the formula means that we want to express q in terms of u.

Since q is being divided by 4 we undo this by multiplying both sides by 4 to get:

$$q = u \times 4$$

Which is clearly the same as "Start with u, multiply by 4, and the answer is q."

or, as we agreed to do,

$$q = 4u$$

Complications begin to set in when it takes more than one arithmetical computation to invert the formula. The point is that we still proceed by retracing our steps and using the "undoing" language.

Let's look at an example that involves two arithmetical steps.

As the number of steps increases, the need for a regular course in algebra also increases. But the meaning of algebra stays the same.

You are ordering cheese from a mail order catalogue. The price is \$4 per pound plus a shipping and handling charge of \$3 per order. You decide to order 5 pounds of the cheese. How much will you have to pay for the order?

Well, each pound will cost $4. So to find the cost of 5 pounds we have to multiply $4 by 5, which gives us $20. Then we have to pay an additional $3 to handle shipping. Therefore we add $3 to $20 and conclude that the order will cost us $23.

In doing this problem, we used a relationship between the number of pounds of cheese we bought and the total cost of the cheese. In particular, the cost of P pounds of cheese will be C dollars and the relationship is:

Step 1:	*Start with P (the number of pounds)*	*In the above illustration P was 5.*
Step 2:	*Multiply by 4 (the cost per pound)*	*5 X 4 = 20*
Step 3:	*Add 3 (the shipping and handling charge)*	*20 + 3 = 23*
Step 4:	*The answer is C (the total cost in dollars)*	*$23*

Example 16

Use the formula below to determine the value of C when P = 16.

(1) Start with P
(2) Multiply by 4
(3) Add 3
(4) The answer is C.

Answer: When P = 16, C = 67

We simply carry out the given steps in the given order, beginning with P = 16.

That is:

(1) Start with P (16)
(2) Multiply by 4 (16 X 4 or 64)
(3) Add 3 (64 + 3 or 67)
(4) The answer is C (67)

While the value of C depends on the value of P, 4 and 3 are constants in this formula. No matter what P is we multiply it by 4 and then add 3.

In terms of our cheese-catalogue illustration this tells us that 16 pounds of cheese will cost us $67.

Sometimes when we order we know how much we can afford to spend and we want to know how much we can buy for this amount. For example suppose we want to order the same cheese but we only had $35 to spend. We'd have to figure out how many pounds we could buy for $35. *This is the inverse of what we did before. Namely originally we started with the number of pounds we bought (P) and computed the total cost in dollars (C). This time we start with the number of dollars we want to spend (C) and we want to compute the number of pounds we can buy for this amount (P).* This involves being able to do the following example.

<u>Example 17</u>

Use the formula below to find the value of P when C = 35

(1) Start with P
(2) Multiply by 4
(3) Add 3
(4) The answer is C

Answer: When C = 35, P = 8

Here the number we're given goes in the last step. This tells us that we have to invert the formula. That is, we have:

(1) Start with P
(2) Multiply by 4
(3) Add 3
(4) The answer is 35.

So we start with Step 4 and retrace our steps *in the exact order as shown,* using again the "undoing" language.

That is:

$$\begin{array}{llll}
(1) & \text{Start with } P & \text{The answer is } P & (8) \\
 & \downarrow & \uparrow & \\
(2) & \text{Multiply by } 4 \rightarrow\rightarrow\rightarrow & \text{Divide by } 4 & (8) \\
 & \downarrow & \uparrow & \\
(3) & \text{Add } 3 \rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow & \text{Subtract } 3 & (32) \\
 & \downarrow & \uparrow & \\
(4) & \text{The answer is } 35 \rightarrow\rightarrow & \text{Start with } 35 & (35)
\end{array}$$

See the "undoing" language? We replaced "multiply" by "divide" and "add" by "subtract"

More generally, the inverse formula is:

(1) Start with C *(C = 35 was just for this example)*

(2) Subtract 3

(3) Divide by 4

(4) The answer is P

Notice the importance of order. The last step we did in the original formula before writing the answer was to add 3. This becomes the first step in the "undoing" process. That is, we got 35 after we added 3, so before we added the 3, we had 32 (that is, 35 - 3)

Note:

It is relatively easy to convert the verbal form into an algebraic formula. For example:

(1) Start with P..............P

(2) Multiply by 4.............4P

(3) Add 3 ....(4P) + 3

(4) The answer is C............C = (4P) + 3

The parentheses tell us that the 3 is added after we multiply P by 4. That is we consider everything within the parentheses to be one number.

If we wanted to undo 35 = (4P) + 3, we see that P is inside the parentheses. To undo adding 3 to 4P we subtract 3 from both sides of the equation to get:

$$\begin{array}{rcl}
35 & = & (4P) + 3 \\
-\ 3 & & -\ 3 \\
\hline
32 & = & 4P
\end{array}$$

Finally since P is being multiplied by 4, we undo it by dividing both sides of the equation by 4 to get:

$$32 \div 4 = P$$

This agrees with the sequence of steps we already used. Namely we start with 35, then subtract 3, then divide by 4 to get 8 as the answer.

Many beginning students get confused as to whether we first should subtract 3 or divide by 4. Thinking of the formula verbally, and proceeding as we did in this example, usually helps to make things clearer.

(4P) + 3 tells us that we first multiply by 4 and then add 3. Since adding 3 was the last thing we did, it is the first thing we undo.

These ideas are discussed more fully in an algebra course. For our purposes we want only to emphasize the close bond between arithmetic and

algebra. So we'll conclude this module by presenting two practical examples in which formulas and the undoing method prove to be useful.

Illustration 1:

The Relationship between Fahrenheit and Celsius Temperature Scales

There are two common systems used for measuring temperature. The English system uses Fahrenheit degrees and the metric system uses Celsius degrees. Because most of us are used to Fahrenheit degrees, we describe the formula for converting Fahrenheit readings to Celsius readings.

You don't have to understand the formula in order to use it. However, in the event you are interested, the derivation of this formula is in the appendix at the end of this module.

Step 1: Start with the Fahrenheit reading, F

Step 2: Subtract 32. $F - 32$

Step 3: Multiply by $\frac{5}{9}$. $\frac{5}{9}(F - 32)$

(If fractions still bother you, remember that multiplying by 5/9 is the same as multiplying by 5 and then dividing by 9)

Step 4: The answer is the corresponding Celsius reading. $C = \frac{5}{9}(F - 32)$

Remember that we don't write $\frac{5}{9} \times (F - 32)$

For example to multiply 36 by 5/9 means to take 5/9 of 36. To do this we can divide 36 by 9 to get 4 and then multiply by 5 to get 20. We could also multiply 36 by 5 to get 180 and then divide by 9 to get 20.

Note:

If we started with $C = \frac{5}{9}(F - 32)$ *and wanted to write the formula more verbally, we'd start with* F *because it is involved in the arithmetic. Then because everything in parentheses is one quantity, we'd next subtract 32. Finally we'd multiply by* $\frac{5}{9}$ *to get* C*; and this is exactly what Steps 1 through 4 tell us.*

Example 18

Use the formula $C = \frac{5}{9}(F - 32)$
to find the Celsius temperature when the Fahrenheit temperature is 68 degrees.

Answer: When $F = 68$, $C = 20$

We sometimes say that 68°F is the same as 20°C

In terms of the formula, F stands for the Fahrenheit temperature. So we replace F by 68 in the formula to get:

$$
\begin{aligned}
C &= \frac{5}{9}(\underset{\downarrow}{F} - 32) \\
&= \frac{5}{9}(68 - 32) \\
&= \frac{5}{9}(36) = \\
&= \frac{5}{9} \times \frac{36}{1} \\
&= \frac{180}{9} \\
&= 20
\end{aligned}
$$

Since 68 - 32 is within the parentheses, we treat it as one number. That is, we're taking 5/9 of (68 - 32). We are not subtracting 32 from 5/9 of 68.

In the more verbal form we have:

(1) Start with F (68)

(2) Subtract 32 (68 - 32 = 36)

(3) Multiply by $\frac{5}{9}$ ($\frac{5}{9} \times 36 = 20$)

(4) The answer is C (20)

The first method is stressed in most algebra courses. Because this is still an arithmetic course, we prefer to emphasize the arithmetic by using the verbal form.

In using the formula that relates Fahrenheit degrees to Celsius degrees, keep in mind that the answer need not be a whole number.

Example 19

Convert 80°F into the Celsius scale.

Answer: $80°F = 26\frac{2}{3}°C$

If you used a calculator the answer may have come out in a form like 26.66667.

The formula remains the same. We start with the Fahrenheit reading, which in this case is 80. Then we subtract 32 to get 58. Then we multiply 58 by $\frac{5}{9}$ to get:

$\frac{5}{9} \times 58 =$

$\frac{5}{9} \times \frac{58}{1} =$

$\frac{240}{9}$ $\quad 26R6 = 26\frac{6}{9} = 26\frac{2}{3}$

$$\begin{array}{r} 26R6 \\ 9\,)\overline{240} \\ -\underline{18} \\ 60 \\ -\underline{54} \\ 6 \end{array}$$

In taking 5/9 of a number it is easier to first divide by 9 if the number is a multiple of 9. If the number is not a multiple of 9 (as is the case with 58) it easier to multiply by 5 first. In this way we don't have to use mixed numbers until the last step.

In algebraic form, we have:

$$\begin{aligned} C &= \frac{5}{9}(80 - 32) \\ &= \frac{5}{9}(58) \\ &= \frac{5}{9} \times 58, \end{aligned}$$

which is an arithmetical equation.

The trickier part is to invert the formula in the case where we're given the Celsius reading (C) and we want to find the Fahrenheit reading (F)

<u>Example 20</u>

What is the Fahrenheit reading if the Celsius reading is 30?

Answer: When C = 30, F = 86

The verbal formula is:

(1) Start with F

(2) Subtract 32

(3) Multiply by $\frac{5}{9}$

(4) The answer is C.

So in this case we have:

$$\begin{array}{lllll} (1) & \text{Start with F} & & \text{The answer is F} \\ & \downarrow & & \uparrow \\ (2) & \text{Subtract 32} & \leftrightarrow\leftrightarrow\leftrightarrow & \text{Add 32} \\ & \downarrow & & \uparrow \\ (3) & \text{Multiply by } \frac{5}{9} & \leftrightarrow\leftrightarrow\leftrightarrow & \text{Divide by } \frac{5}{9} \\ & \downarrow & & \uparrow \\ (4) & \text{The answer is 30} & \rightarrow\rightarrow & \text{Start with 30} \end{array}$$

Recall that dividing by 5/9 is the same as multiplying by 9/5. That is, to divide, we invert and multiply.

In other words, the inverse formula is:

(1) Start with C

(2) Divide by $\frac{5}{9}$ (or Multiply by $\frac{9}{5}$)

(3) Add 32

(4) The answer is F.

Remember the order. Since subtracting 32 was the first arithmetic step in the original formula, subtracting 32 will be the last arithmetic step in the inverse formula.

Starting with C = 30 we have:

(1) Start with 30

(2) Multiply by $\frac{9}{5}$ ($\frac{9}{5}$ X 30 = 9 X 6 = 54)

(3) Add 32 (54 + 32 = 86)

(4) The answer is C (86)

<u>Note</u>

In the language of algebra, we have

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ &\downarrow \\ 30 &= \frac{5}{9}(F - 32) \end{aligned}$$

which is an algebraic equation. To isolate F we have to unblock the parentheses first. Since 5/9 is multiplying the parentheses, we first divide both sides of the equation by 5/9 (or multiply by 9/5) to get:

$$\frac{9}{5} \text{ X } 30 = F - 32$$

or

$$54 = F - 32$$

To get F by itself we then undo subtracting 32 by adding 32 to both sides of the equation to get:

$$\begin{array}{r} 54 = F - 32 \\ + \underline{32 \qquad + 32} \\ 86 = F \end{array}$$

Again, remember that if this approach is too abstract, it simply does in symbols what we did above in verbal form.

Additional drill is left for the Self Test, but for now let's turn our attention to a final illustration.

Illustration 2:

The Relationship Between the Time and the Distance that a Freely-Falling Body Falls

In the 16th century the Italian scientist Galileo discovered a relationship between the distance a freely-falling body fell and how long it took to fall that distance. Verbally the formula is:

(1) Start with the time t in seconds.

(2) Square the time

(3) Multiply by 16

(4) The answer is the distance d in feet.

This formula is valid in the absence of air resistance and relatively close to the surface of the earth. Again you don't have to understand why the formula works in order to be able to use the formula.

Algebraically the formula is

(1) Start with t

(2) Square it.......t^2

(3) Multiply by 16......$16(t^2)$

(4) The answer is d in feet.....$d = 16(t^2)$

The parentheses indicate that we multiply by 16 after we square t. That is, we do what's in the parentheses first.

Example 21

Use the formula $d = 16(t^2)$ to find d if $t = 3$.

Answer: When $t = 3$*,* $d = 144$

If we replace t by 3 we get:

$$d = 16(3^2)$$

$$= 16(9)$$

$$= 16 \times 9 \text{ or } 144$$

In other words, the freely-falling body falls 144 feet during the first 3 seconds it falls.

In words:

(1) Start with 3

(2) Square it (9)

(3) Multiply by 16 (9 X 16 = 144)

(4) The answer is d feet (144 feet)

In using formulas it is important to remember denominations.

Example 22

Use the formula $d = 16(t^2)$ to find how far an object falls in 1 minute.

Answer: In 1 minute the object falls 57,600 feet

To use the formula t must be in seconds. Therefore if we replace t by 1, we are finding the distance the object falls in 1 second. Since we want the distance it falls in 1 minute, we have to use the fact that a minute is 60 seconds.

That is:

(1) Start with the time in *seconds*.......60

(2) Square it........$60^2 = 60 \times 60 = 3,600$

(3) Multiply by 16....3,3600 X 16 = 57,600

(4) The answer is the number of feet the object falls.............57,600 feet

Algebraically we have:

$$d\ (feet) = 16(60^2) = 16(3,600) = 57,600$$

The algebra comes into play when we want to invert the formula. We would do this, for example, if we knew how far the object fell and we wanted to determine how long the object had been falling.

Example 23

Given the formula $d = 16(t^2)$, find the value of t when d = 784.

Answer: 7

That is, it takes a freely-falling body 7 seconds to fall 784 feet.

We want to invert the formula:

(1) Start with t

(2) Square it

(3) Multiply by 16

(4) The answer is d

We want to replace d by 784 and then figure out what the value of t was.

Remembering that the square root undoes squaring and that division undoes multiplication, we have:

$$\begin{tikzcd}
\text{(1) Start with } t \arrow[d] & \text{The answer is } t \\
\text{(2) Square it} \arrow[d] \arrow[r] & \text{Take the square root} \arrow[u] \\
\text{(3) Multiply by 16} \arrow[d] \arrow[r] & \text{Divide by 16} \arrow[u] \\
\text{(4) The answer is } d \arrow[r] & \text{Start with } d \arrow[u] \\
 & \textit{The inverse formula} \arrow[u]
\end{tikzcd}$$

Hence the inverse formula in verbal form is:

(1) Start with d............784

(2) Divide by 16............$784 \div 16 = 49$

(3) Take the square root......$\sqrt{49} = 7$

(4) The answer is t 7

If we wanted to use the algebraic form of undoing, we start with $784 = 16(t^2)$. Since t is inside the parentheses and 16 is multiplying what's in the parentheses, we begin by dividing both sides of the equation by 16 to get $784 \div 16 = t^2$. Then since t appears squared we undo this by taking the square root of both sides to get:

In fraction form; $\frac{784}{16} = \frac{\cancel{16}(t^2)}{\cancel{16}}$

$$\sqrt{784 \div 16} = t$$

$$\sqrt{49} = t$$

$$7 = t$$

This is an arithmetical equation that can be solved by what we've done in the previous eleven modules.

Pursuing these topics further would best be served as part of an algebra course. For this reason, additional remarks will be reserved for the Self Test.

This completes our saga of arithmetic. We began with the simple observation of how people learned to count and proceeded to show how special language made it easy to express numbers.

We then showed how logic could help us find numerical answers to problems more effectively than trial-and-error randomness.

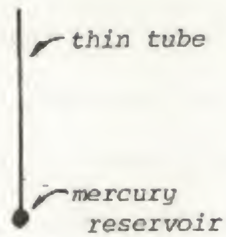
Once we learned how to do various arithmetical computations, we showed how they were used in solving problems that occurred in everyday life. This led to our being able to construct and interpret formulas.

This is probably the most difficult part, but if we can't do this there is not much use in being able to do the computations.

Finally, we saw that there were times when even if we had the formula we might have to know how to invert it in order to get the correct answer to a problem. Perhaps the most important thing about being able to invert a formula is the high degree of logic that's involved. In the long run, the most valuable thing any course has to offer is to help us learn to think.

So if you feel more comfortable with numbers and arithmetic now; if you feel that you are better able to think in quantitative terms; if you feel you are now willing to try the next level mathematics course; or most importantly, if you feel better about yourself now than when the course began; then we feel that the course has been an unqualified success.

APPENDIX: A Note on Temperature Scales



Heating a liquid will usually make it expand; cooling it will make it contract. This is the principle behind a mercury thermometer. Mercury is placed inside a tube with a very narrow cross section. In this way if the mercury expands it has no room to move, other than up the tube; and if it contracts, it has no room to move, other than down the tube. A bulb at the bottom of the tube serves as a reservoir for the mercury. The cross section of the tube is so small that we usually encase it in a magnifying glass to help us better read the mercury level (see diagrams in the margin).

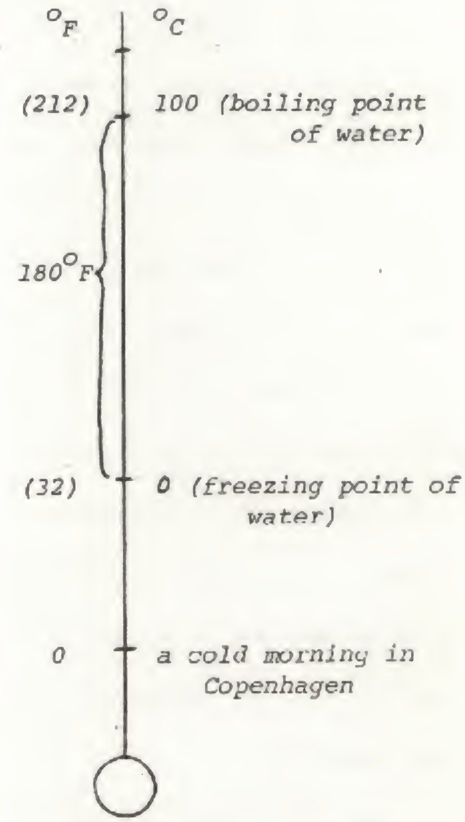
The magnifying glass makes it easier for us to see the mercury level.

Both Fahrenheit and Celsius used a scale of 100 to base their readings on. Fahrenheit left the thermometer outside his window one cold morning and marked the mercury level at that time by 0. Then he took his body temperature and marked the mercury level at that time by 100. *(So if things had been "normal", normal body temperature would have been 100°F)*

The word "grad" means "degree" and "centi" means "hundred". So "centigrade" (which was used before the term "Celsius) simply meant "100 degrees).

Celsius wanted reference points that were less subjective that Fahrenheit's. After all what was noteworthy about the weather one morning where Fahrenheit lived or about Fahrenheit's body temperature .

Since everyone could relate to the freezing point and boiling point of water, Celsius called the freezing point of water 0 ($0^{o}C$) and the boiling point 100 ($100^{o}C$). In terms of Fahrenheit's scale it turned out that the freezing point of water was 32 ($32^{o}F$) and the boiling point was 212 (212^{o}).



The key point is that the mercury level doesn't depend on whether we use the Fahrenheit scale or the Celsius scale. All that changes is the number (adjective) that names the number of degrees. For example, $100^{o}C$ and $212^{o}F$ name the same mercury level (temperature).

If we measure the change in mercury level in terms of Celsius's scale in going from the freezing point of water ($0^{o}C$) to the boiling point ($100^{o}C$) we get $100^{o}C - 0^{o}C$ or $100^{o}C$. If we use Fahrenheit's scale in going from the freezing point of water ($32^{o}F$) to the boiling point of water ($212^{o}F$) we get $212^{o}F - 32^{o}F$ or $180^{o}F$. *That is, there are:*

$100^{o}C$ *per* $180^{o}F$

or

$5^{o}C$ *per* $9^{o}F$

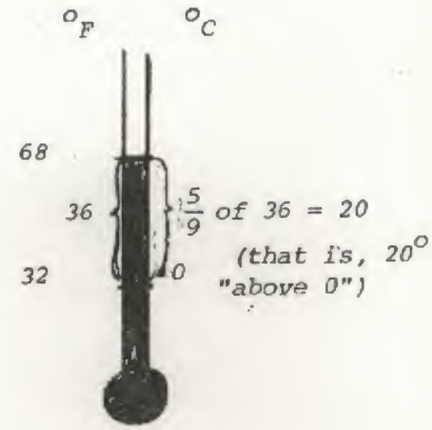
We're divding both numbers in the ratio by 20, which doesn't change the rate.

If we now divide both sides of the last rate by 9 we get:

$\frac{5}{9}\,^{o}C$ *per* $1^{o}F$

That is, 5/9 of the change in Fahrenheit degrees gives us the change in Celsius degrees.

So suppose we're given a Fahrenheit reading and want to see how it compares with $0^{o}C$. For example, suppose we had the reading $68^{o}F$. $0^{o}C$ corresponds to $32^{o}F$. So to see how high the mercury is above $0^{o}C$, we begin by subtracting $32^{o}F$ from $68^{o}F$. This gives us the height above $0^{o}C$ measured in the Fahrenheit scale ($68^{o}F - 32^{o}F = 36^{o}F$). Since every $9^{o}F$ is equivalent to $5^{o}C$, we take $\frac{5}{9}$ of $36^{o}F$ to get $20^{o}C$.



We didn't have to start with $68^{o}F$. In fact, from a more general point of view:

1. Start with the Fahrenheit reading (F)
2. Subtract 32 (to get the height scaled to $0^{o}C$)
3. Multiply by $\frac{5}{9}$ (to convert the answer to ^{o}C)

In terms of a formula:

$$(F - 32) \times \frac{5}{9} = C$$

or

$$C = (F - 32) \times \frac{5}{9}$$

$$= \frac{5}{9} \times (F - 32)$$

$$= \frac{5}{9}(F - 32)$$

Notice that it is not important to understand why the formula works in order to be able to use the formula. However it is "enriching" to understand why formulas work. We'll talk more about this in Video Lecture 12B to help make sure you understand this important point.